UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 10-41
Statistics and Econometrics Series 25
October 2010

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COMPARING SAMPLE AND PLUG-IN MOMENTS IN ASYMMETRIC GARCH MODELS.

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Abstract

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Keywords: Autocorrelations of squares, Cross-correlations, heterocedasticity, kurtosis, leverage effect, TGARCH.

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[3] We acknowledge financial support from the Spanish Government project ECO2009-08100.

Comparing sample and plug-in moments in asymmetric GARCH models.

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JEL: C22

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1. INTRODUCTION

Conditionally heterocedastic time series are characterized by having excess kurtosis and positive autocorrelations of squares. Furthermore, in the presence of leverage effect, the cross-correlations between the series of returns, y_t, and y_{t+k}^2 are negative. It is very common to analyze the adequacy of a fitted model by comparing its implied or plug-in kurtosis, autocorrelations of squares and cross-correlations with the corresponding sample moments of the original returns; see Karanasos and Kim (2006) and Figà-Talamanca (2008) among many others. However, although the finite sample properties of the sample kurtosis and autocorrelations of squares have already been analyzed, those of the corresponding plug-in moments are unknown. In this paper, we fill up this gap. Furthermore, we study whether comparing plug-in and sample moments is appropriate when analyzing the adequacy of a fitted model. We focus on the TGARCH model of Zakoïan (1994) for its good performance when representing heterocedastic time series with leverage effect; see Rodríguez and Ruiz (2009).

2. FINITE SAMPLE PROPERTIES OF PLUG-IN AND SAMPLE MOMENTS

The TGARCH(1,1) model is given by

$$\begin{aligned} y_t &= \varepsilon_t \sigma_t \\ \sigma_t &= \omega + \alpha \left| y_{t-1} \right| + \beta \sigma_{t-1} + \delta y_{t-1} \end{aligned} \tag{1}$$

where ε_t is a serially independent sequence with zero mean, variance one and symmetric density and σ_t is the volatility. The distribution of ε_t is assumed to be either Gaussian or Student-7. The parameters of model (1) have to be adequately restricted to guarantee stationarity, finite fourth order moment of y_t and positive conditional variances; see, for example, Rodríguez and Ruiz (2009).

In order to compare the finite sample properties of the plug-in and sample moments we generate $R = 1000$ series of sizes $T = 500$, 2000 and 5000 by the TGARCH model with parameters $\alpha = 0.17$, $\beta = 0.8$ and $\delta_T = -0.1$[1]. The parameter ω is such that the marginal variance of y_t is one. Denote by κ, $\rho_2(1)$ and $\rho_{12}(1)$, the population kurtosis, first order autocorrelation of squares and first order cross-correlation between y_t and y_{t+1}^2, respectively which are given by $\kappa = 9.01$,

[1] Results for other specifications as EGARCH, TGARCH, GJR and QGARCH models are available from the authors upon request. The conclusions are similar regardless the model or whether there is or not leverage effects.

$\rho_2(1) = 0.344$ and $\rho_{12}(1) = -0.112$ when the errors are Gaussian, whereas $\kappa = 16.91$, $\rho_2(1) = 0.237$ and $\rho_{12}(1) = -0.077$ when they are Student-7. The corresponding plug-in moments are denoted by $\hat{\kappa}$, $\hat{\rho}_2(1)$ and $\hat{\rho}_{12}(1)$. Finally, the sample moments are denoted by k, $r_2(1)$, $r_{12}(1)$. For each time series generated, we compute the sample and the plug-in moments[2]. Table 1 reports their Monte Carlo relative biases and standard deviations.

Consider first the results for the kurtosis. The plug-in kurtosis have positive relative biases which can be very large when $T = 500$. Both, biases and standard deviations, decrease with the sample size. However, the relative biases of the sample kurtosis are negative and very large regardless of the error distribution and sample size considered; see An and Ahmed (2008) for a similar conclusion. Furthermore, the biases hardly decrease with the sample size. The relative biases of $k - \hat{\kappa}$ are rather large and even larger when the errors are Student-7. Therefore, the sample and plug-in kurtosis tend to be far apart even when the model is correctly specified; see also the first column of Figure 1 which plots kernel densities of k, $\hat{\kappa}$ and their differences for T=2000 and Gaussian errors. Comparing plug-in and sample kurtosis may lead to misleading conclusions about the adequacy of a fitted model.

When looking at the results corresponding to the plug-in first order autocorrelation of squares, we can observe that the relative biases are negative. The magnitude of the biases and standard errors are similar regardless of the error distribution. On the other hand, although the biases of the sample autocorrelations are also negative, they are much larger in magnitude; Bollerslev (1988), He and Teräsvirta (1999) and Pérez and Ruiz (2003) also report negative biases of the sample autocorrelations. As expected, both the biases and standard deviations decrease with the sample size. Therefore, we expect that the plug-in first order autocorrelations of squares would be in average larger than their sample counterparts and, obviously, closer to the population autocorrelations. Also note that the standard deviations of the sample autocorrelations are much larger than those of the plug-in. Consequently, as in the case of the kurtosis, comparing the plug-in first order autocorrelation of squares with the sample autocorrelation can lead to reject the adequacy of a well specified GARCH model; see also the second column of Figure 1 which plots kernel densities of $r_2(1)$, $\hat{\rho}_2(1)$ and their differences when $T = 2000$ and the errors are Gaussian.

Finally, the relative biases and standard deviations of the sample cross-correlations depend on the error distribution and sample size considered. It is also important to note that although, the

[2]The parameters have been estimated by ML using software developed by the first author in Matlab.

biases of the sample cross-correlations have magnitudes larger than those of the corresponding plug-in cross-correlations, they are, in general, relatively small. Once more, the standard deviations of the differences are very large compared with the magnitude of the cross-correlations. Therefore, comparing $\hat{\rho}_{12}(1)$ and $r_{12}(1)$ may also be rather misleading to conclude about the adequacy of an asymmetric GARCH model fitted to a given time series of returns; see the third column of Figure 1.

3. EMPIRICAL APPLICATION

In this section we fit the TGARCH model to series of daily returns of the SP500 index and of the EUR/USD exchange rate observed from January 2nd 2002 to June 25th 2010. Figure 2 plots both series together with their corresponding sample autocorrelations of squares and cross-correlations between y_t and y_{t+h}^2. The autocorrelations of squares of both series are significant; see also Table 2 which reports the corresponding Box-Ljung statistic. The cross-correlations of SP500 returns are also significant and negative suggesting the presence of leverage effect. However, the cross-correlations of EUR/USD returns are not significant. Therefore, a GARCH model with leverage effect may be appropriate for the SP500 returns while the EUR/USD returns could be represented by a symmetric GARCH model. We fit the TGARCH model with Student-ν errors to each of these series. The estimated model for the SP500 returns is given by

$$\sigma_t = \underset{(0.002)}{0.012} + \underset{(0.009)}{0.054}\,|y_{t-1}| + \underset{(0.008)}{0.948}\sigma_{t-1} \underset{(0.006)}{-0.054}\,y_{t-1}$$

with $\widehat{\nu} = 11.95$. The estimated volatility of the EUR/USD returns is given by

$$\sigma_t = \underset{(0.002)}{0.003} + \underset{(0.007)}{0.040}\,|y_{t-1}| + \underset{(0.007)}{0.963}\sigma_{t-1} \underset{(0.005)}{-0.004}\,y_{t-1}$$

with $\widehat{\nu} = 15.08$. Note that, as expected, the asymmetry of the EUR/USD returns is not significant. Table 2, which reports several moments of the standardized returns, shows that they have smaller kurtosis than the original observations. Furthermore, when looking at the Box-Ljung statistic to test for the significance of the autocorrelations of squares and cross-correlations, we can observe that they are not any more significant. Therefore, it seems that the TGARCH model is able to explain the autocorrelations of squares and cross-correlations between returns and future squared returns.

Finally, Table 2 reports the plug-in moments obtained after substituting the parameter estimates in the expressions of the corresponding population moments. When looking at the results for the SP500 returns, we observe that the plug-in kurtosis is much larger than the sample kurtosis. Therefore, we may think that the TGARCH model is not adequate to represent the SP500 kurtosis. However, according to our simulation results, the plug-in kurtosis is positively biased while the sample kurtosis has a negative bias. Therefore, in spite of the large distance between the sample and plug-in kurtosis, the TGARCH model could still be adequate for the SP500 returns. When comparing the plug-in and sample autocorrelations of squares and cross-correlations between returns and future squared returns, we can observe that the differences are pretty small; see also Figure 2 where the plug-in correlations have been plotted together with the corresponding sample correlations. However, although the sample autocorrelations of squares are larger than the plug-in autocorrelations, which is in contrast with the bises observed in our Monte Carlo results, remember that the dispersion of the differences between sample and plug-in autocorrelations is very large. When comparing plug-in and sample moments of the EUR/USD returns, we can observe that all moments are very similar.

4. CONCLUSIONS

This paper analyses the suitability of comparing plug-in and sample kurtosis, autocorrelations of squares and cross-correlations between returns and future squared returns when checking the adequacy of a fitted GARCH model. We show that the biases of the sample and plug-in kurtosis have opposite sign. The differences between sample and plug-in autocorrelations and cross-correlations have very large dispersions. Therefore, comparing both quantities is not adequate.

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Table 1.- Monte Carlo relative biases and standard deviations (in parenthesis) of sample and plug-in moments and their differences.

	Gaussian			*Student-7*		
	T=500	*T=2000*	*T=5000*	*T=500*	*T=2000*	*T=5000*
$\hat{\kappa}$	37.89% (14.448)	19.06% (8.169)	4.76% (2.466)	17.38% (28.600)	15.31% (14.52)	13.25% (9.187)
k	−35.67% (3.540)	−24.41% (3.409)	−20.52% (2.944)	−56.56% (6.261)	−42.09% (5.692)	−34.65% (5.216)
k-$\hat{\kappa}$	−77.11% (14.027)	−43.62% (3.409)	−25.29% (2.943)	−65.56% (21.559)	−57.41% (14.106)	−47.90% (9.518)
$\hat{\rho}_2(1)$	−6.80% (0.061)	−1.32% (0.034)	−0.46% (0.022)	−8.00% (0.063)	−2.07% (0.086)	−0.24% (0.022)
$r_2(1)$	−27.67% (0.109)	−15.32% (0.085)	−11.85% (0.067)	−26.66% (0.106)	−11.69% (0.033)	−8.04% (0.072)
$r_2(1) - \hat{\rho}_2(1)$	−22.85% (0.096)	−14.13% (0.085)	−11.40% (0.067)	−18.81% (0.091)	−9.17% (0.084)	−7.78% (0.069)
$\hat{\rho}_{12}(1)$	−5.47% (0.023)	−1.90% (0.012)	−0.08% (0.007)	−1.72% (0.022)	−1.66% (0.011)	−1.80% (0.007)
$r_{12}(1)$	−2.18% (0.084)	4.60% (0.047)	3.70% (0.033)	19.93% (0.079)	22.81% (0.051)	23.19% (0.036)
$r_{12}(1) - \hat{\rho}_{12}(1)$	3.30% (0.096)	6.92% (0.047)	4.52% (0.033)	21.02% (0.071)	24.41% (0.049)	24.93% (0.039)

Figure 1.- Kernel densities of the Monte Carlo sample moments (dashed), plug-in moments (continuous) (top panel) and their differences (lower panel). The vertical line represents the population moments. The first column corresponds to κ_y, the second to $\rho_2(1)$ and the third to $\rho_{12}(1)$.

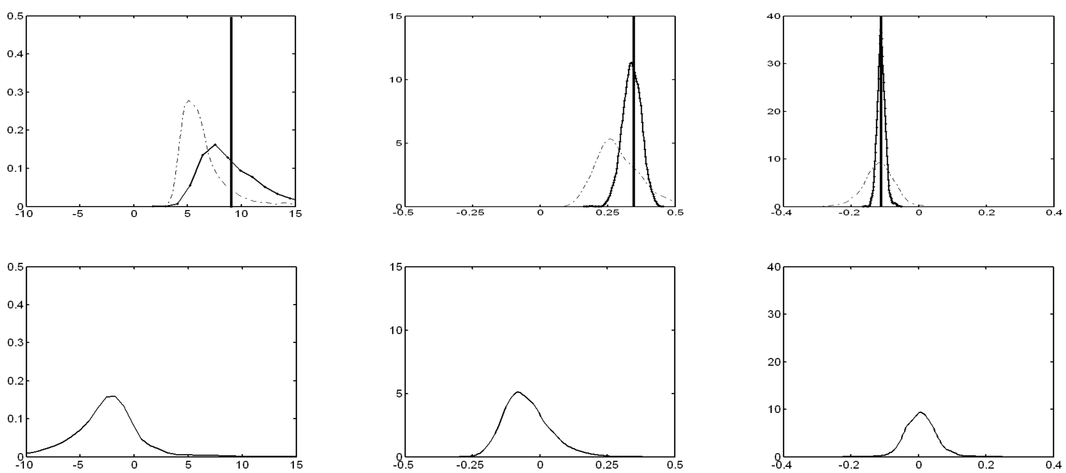
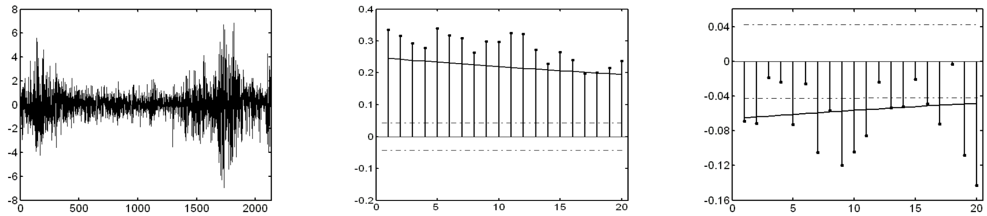


Figure 2.- Daily returns (first column), sample autocorrelations of squares (second column) and cross-correlations between y_t and y_{t+1}^2 (third column) together with 95% confidence intervals (discontinuous line) and plug-in moments (continuous line).

SP500



EUR/USD

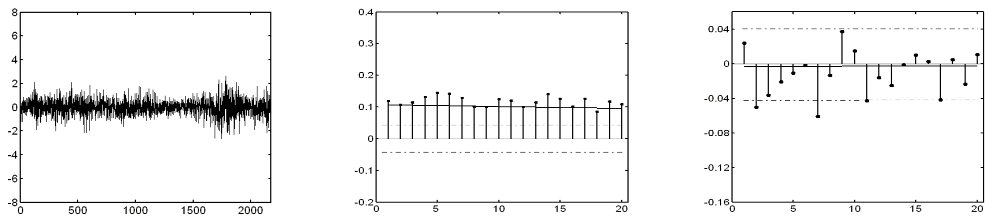


Table 2.- Sample moments together with corresponding diagnostic statistics and plug-in moments.

	SP500			EUR/USD		
	Sample	Residuals	Plug-in	Sample	Residuals	Plug-in
$Kurtosis$	7.12	4.19*	16.82	4.28	3.50*	4.71
$\rho_2(1)$	0.34	-0.07	0.25	0.12	-0.03	0.11
$Q_2(20)$	3148.50*	30.04	–	616.08*	15.86	–
$\rho_{21}(1)$	-0.07	-0.05	-0.07	0.02	0.02	0.00
$Q_{21}(20)$	217.15*	30.77	–	34.41	18.30	–

* Significant at 5% level